Behavioral Heterogeneity in Large Economies

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Abstract: Grandmont's ([14]) notion of behavioral heterogeneity is reformulated in a non parametric set-up such that the space of budget share functions admits a "uniform" probability distribution. If the population is distributed according to this measure, the aggregate budget share function is constant with respect to changes in prices and income. This exact insensitivity of the market budget share function is known to imply uniqueness and global stability of any competitive equilibrium. Here, it is not explained by any insensitivity property at the micro-economic level, but rather by a perfect "balancing effect". Eventually, it is proved that the insensitivity property holds *approximately* for a *finite* population sufficiently close to, but distinct from, the perfectly heterogenous one.

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1. Introduction

One of the main challenges, both of macro-economics and econometrics, is to provide micro-economic foundations for its analysis of aggregate demand. It is well known however that the sole restrictions induced by individual optimization on aggregate excess demands are essentially continuity, zero-homogeneity, Walras' identity and a boundary condition — this is the celebrated Sonnenschein-Mantel-Debreu theorem, see, e.g., [12]. Therefore, if one abstracts the boundary condition, the vector field induced, say on the unit sphere of normalized prices, by aggregate demand is locally arbitrary. As a consequence, the set of equilibrium prices of an exchange economy is, in general, not unique, and, in fact, essentially arbitrary (cf. [27]). More recently, these results have been extended to excess demand functions in economies with incomplete markets ([6]) and to demand functions both with complete and incomplete markets ([9], [10], [7], and more recently [30]). The main economic lesson of this is essentially negative: individual optimization does not sufficiently restrict the aggregate demand, at least locally, to get sound properties such as the 'Law of Demand', gross substitutability, and thereby uniqueness and stability. Consequently, so goes the story, general equilibrium theory is often viewed as unable to make any observable, predictive statement while one of its favorite exercises — comparative statics — relies on especially vulnerable grounds. As was suggestively expressed by [23], one has to confess that "the emperor has no clothes".[2]

The decisive contribution of Hildenbrand ([19]) brought some hope by showing that certain restrictions on the distribution of income can induce macro-economic properties such as the "Law of Demand", even if these properties are not satisfied at the micro-economic level.[3] Though, as underlined by the author, these restrictions — in particular the fact that the income distribution was assumed to be downward slopping and the collinearity of initial endowments to generate uniqueness and stability under the Walras' tatonnement of the price equilibrium in exchange economies — are not very realistic, this contribution induced a shift of viewpoint.[4] Hence, an important issue became: What are the properties of the mean demand induced by a large, heterogenous population of possibly irrational and/or irregular households ? Grandmont ([14]) gave a first, illuminating answer to this question by proving that, within a parametric model of demand functions, sufficiently dispersed demand functions may generate the diagonal dominance of the Jacobian of market demand. Quah ([31]) extended Grandmont's formalism to allow for the possibility of atoms, and considered situations involving a weaker heterogeneity assumption on individual demand functions. He replaced the assumption that incomes do not depend upon prices by the requirement that the distributions of individual demand functions and income are independent, and eventually derived the uniqueness and stability under the Walras' tatonnement of the price equilibrium in exchange and production economies — without any collinearity of initial endowments. Finally, [24] extended this approach to a non-parametric setting. The main common idea behind these various frameworks is to show that enough heterogeneity of behavior can explain the insensitivity of the market budget share function to changes in prices and/or income. For this purpose, one considers some well-defined metric on, say, $\big(\mathcal{W},\nu\big)$, the probability space of household budget share functions, and one analyses some distance-preserving transformations T on the space $\mathcal{W}$. The probability measure ν is then said to satisfy a "high degree" of heterogeneity if the probability of all sets A and $T(A) \subset \mathcal{W}$ is extremely close, whenever T is not too far from the identity mapping. The main piece of good news is then that a "highly heterogenous" population of consumers effectively admits a market budget share function *approximately* insensitive to changes in prices and/or income.[5]

The main message of the present paper is the following : Given some conditions over the space $\mathcal{W}$, it is possible, without any individual rationality assumption, to prove that there exist "uniform" distributions over the space $\mathcal{W}$ such that the aggregate budget share function is *exactly* constant. In other words, for a perfectly heterogenous population, the market takes on exact Cobb-Douglas properties, although no individual behaviour satisfies even the weakest form of rationality or regularity. When framed in a general equilibrium setting, this result implies the uniqueness and global stability of the equilibrium price.

At first glance, it may seem that this is a rather technical issue. Recent debates on behavioral heterogeneity show, on the contrary, that such a result has a clear-cut economic relevance. There are, indeed, two alternative views regarding the precise nature of "behavioral heterogeneity" in Grandmont's model and its successors. According to one view, approximate Cobb-Douglas behavior holds on average in such models because, at any price vector, all but a small fraction of the households do not deviate significantly from Cobb-Douglas behavior. Another view is that Cobb-Douglas behavior arises on the average because agents respond heterogenously to price changes, some by increasing the budget share on a good, others by reducing it, so that average shares remain approximately the same.[6] From an economic standpoint, however, the second phenomenon seems much more appealing than the first, as the first one means that Cobb-Douglas behavior on the average is a foregone conclusion induced by the built-in hypothesis that almost every individual is Cobb-Douglas. Is it possible to show, within his framework, that Grandmont's striking conclusion results solely from some "balancing effect"? The answer, unfortunately, must be 'no', as shown by [3] and [17]. The trouble, indeed, is that, if one pushes Grandmont's argument to its extreme logical consequences, then one is led to a situation that does *not* look like a heterogenous population — quite on the contrary !

This point may be illustrated as follows. Suppose you parameterize the space of budget share functions of your population by some number in $\mathbb{R}$. (This will be the case, for instance, if one considers homothetic transformations *à la* Quah ([31]) acting on demand functions as follows: If f is some generating demand function, each agent in the economy has a demand function f_α, for some $\alpha \in \mathbb{R}$ — where $f_\alpha(p,x) := e^\alpha f(p, e^{-\alpha}x)$, when p is a price vector and x an income level.) Assume, furthermore, that the transformations with respect to which you want to test the 'heterogeneity' of the population you face can be reduced to some collection of translations of the parameter $\alpha \in \mathbb{R}$. Claiming that the population is "highly heterogenous" amounts to assuming that the distribution tends to be invariant with respect to this collection of translations, which means, *that in the limit*, the distribution of agents should converge to some 'uniform' probability distribution on $\mathbb{R}$. Since, however, there is no such probability distribution on the real line, this implies that the measure towards which the distribution of characteristics converges is a measure, on the completed real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, whose support reduces to $\{+\infty, -\infty\}$). In particular, any compact subset of $\mathbb{R}$ is asymptotically of measure zero.

However particular and simple-minded this example might be, it shows the essence of what is going on. Due to this concentration phenomenon, the approximate insensitivity of market budget share, obtained for a highly heterogenous population, can hardly be interpreted: Does it emerge from highly heterogenous reactions of households or, on the contrary, from the insensitivity of almost all (approximately identical) households? In fact the two cases emerge in Grandmont's formalism depending on the boundary behavior of the generating demand function. If this behavior is such that the associated budget share function admits limits on the boudaries of the price-income space, then as brought out by [3] and [17] the "uniform" probability puts all its mass on Cobb-Douglas behaviors.[7] Furthermore, [4] and [25] point out that [24] encounters essentially the same stumbling block.

In this paper, since *exact* insensitivity of the market budget share function is obtained, short of *everybody* being Cobb-Douglas, some agents must increase their share and others must decrease it. In other words, the insensitivity of the budget share function we shall obtain in the aggregate is not explained by any (even approximate!) insensitivity property at the micro-economic level but rather by a perfect "complementary" or "balancing effect"[8]. Moreover, we show that, given any family of individual budget shares, there exists a distribution which has the following much stronger property: every non-empty, open subset of the family is of positive measure. This insures that our 'uniform' measure cannot take its support in, say, a subset of people who would, by chance, react heterogenously, while neglecting the rest of the population. A first step in this direction was made by [25]. The author introduces a new class of distance preserving transformations that ensures that the concentration phenomenon cannot emerge in any orbit induced by a given budget share function. However, since any orbit might have a measure zero, this result does not prevent the concentration phenomenon over the whole space $\mathcal{W}$. Finally, we provide sufficient conditions guaranteeing that our 'uniform' probability distribution is unique.

Of course, to assume that the distribution of characteristics of a given population is uniform (in the precise sense given to this term in this paper) is probably heroic. It should be understood as an "ideal limit-case", like the *continuum* hypothesis in [1]. Our contention is that it proves that behavioral heterogeneity makes sense, even *in the limit*. On the other hand, by analogy with the core equivalence, the 'large' case should prove to be the limit of the finite setting. But this is exactly the way our proof goes. Indeed, rather than reducing the problem to a fixed-point theorem, we explicitly construct a sequence of probabilities with finite support converging to the 'uniform' measure, and for which the aggregate budget share function is approximately insensitive.[9] Hence, the insensitivity property holds approximately for a finite population sufficiently close to, but distinct from, the perfectly heterogenous one (see our Corollary 1, which is the main finding of this paper).

In a somewhat similar context, [13][10] prove that aggregation has a smoothing effect on the demand behavior in a fashion that looks very much like ours. Interpreting a price as a linear operator on the commodity space, they define an action of the group of normalized prices on individual preferences; the notion of "price-dispersed preferences" is then defined by requiring that the distribution on the functional space of smooth utilities be absolutely continuous with respect to the Haar measure on the group. By comparison, the framework employed in this paper ensures that the aggregate budget share function is constant ; this trivially implies that the market demand is differentiable, but it also says much more.[11] The price to pay, however, is that we cannot content ourselves with the absolute continuity with respect to some 'uniform' distribution: we need the distribution of households' characteristics itself to be (approximately) 'uniform' in a certain sense.

In the next section, we set the framework and state our results. We shall be careful when relating our hypotheses to the usual understanding of a "large", "dispersed" and "heterogenous" population. That section concludes with an example. Finally, section 3 contains the proofs.

2. Towards insensitive aggregate budget shares

2.1. The problem

Consider[12] an economy with $L \geq 1$ commodities. Each household is characterized by a *demand function* f:

$$f : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_{+}, \tag{1}$$

which associates to each pair (p, x) of prices and income, a point in the consumption set. As convincingly argued by [24], it is more convenient to work with the corresponding *budget share* function $w : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to [0, \gamma]^L$ where $\gamma > 0$, defined by:

$$\forall (p, x) \quad w(p, x) = \frac{p \quad f(p, x)}{x}. \tag{2}$$

There is obviously a one-to-one and onto relationship between the mapping f and its associated budget share function w.

We consider a subpopulation of households with identical income. Households diverge in their budget share functions, hence in their characteristics affecting demand independently of prices and income. Let denote by $\mathcal{W}$ the space of budget share functions of the economy at hand, endowed with the sup-norm $\|\cdot\|_\infty$. The joint distribution of households' characteristics induces a distribution ν of budget share functions on $\mathcal{W}$. The assumption that all households have the same income is common to all the previous literature, and could be relaxed. Indeed, one easily sees that the properties obtained below for the aggregate budget share of a given subpopulation are preserved through aggregation. Hence, subsequent analysis could apply to suitable sub-economies populated by individuals with identical incomes.

The aggregation problem consists in asking whether there exists certain restrictions on $\mathcal{W}$ and a Borel probability *distribution* ν such that certain properties (e.g., the Law of Demand) are fulfilled by the aggregate budget share function

$$(p, x) \mapsto W(p, x) := \int_{\mathcal{W}} w(p, x)\nu(dw). \tag{3}$$

In other words, we want to take the space $\mathcal{W}$ itself as *given*, provided it belongs to a convenient class of functional spaces, and to prove that an adequate choice of the distribution of households' characteristics, which can be interpreted as representing a perfectly heterogenous population, can induce *per se* economically sound properties on the macro-economic level. In this sense, we view the approach taken in this paper as quite distinct from the one adopted, e.g., by [5]. There, it is argued, loosely speaking, that, given a budget share function it is always possible to construct a complementary one such that their sum satisfies the Law of Demand.

The celebrated "Law of Demand" can be expressed in terms of the aggregate budget share function:

$$\forall p, q \in \mathbb{R}^L_{++}, \quad (p - q) \cdot \Big(p^{-1} \quad W(p, x) - q^{-1} \quad W(q, x)\Big) \square 0. \tag{4}$$

What kind of behavior can be expected from the aggregate budget share function of a large, heterogenous population ? The most demanding property is certainly the insensitivity of the map W with respect to changes in prices and/or income. This property (which is the Cobb-Douglas functions' benchmark) induces, indeed, most of the properties one could dream of: the Law of Demand (since the above inequality clearly holds when W is constant), but also the gross substitutability property, and eventually the uniqueness and global stability (for the Walrasian tâtonnement) of the equilibrium of a pure exchange economy.

2.2. The results

In order to formally define heterogeneity of households with respect to a 'perturbation' of the price-income vector, several transformations on the functional space $\mathcal{W}$ have been proposed in the literature. One can use, for example, as [14] and [24], the *affine* transformations, T_Δ, defined by:

$$\forall w \in \mathcal{W}, \forall \Delta \in \mathbb{R}^{L+1}_{++}, \forall (p,x) \in \mathbb{R}^{L}_{++} \times \mathbb{R}_{++} \quad T_\Delta[w](p,x) = w(\Delta \quad (p,x)). \quad (5)$$

Notice that [31] restricts himself to a smaller class of transformations, called *homothetic* transformations. Denote by $\mathcal{T}$ the class of affine transformations T_Δ. This class verifies the two following conditions:

A) The map $w \mapsto T_\Delta[w]$ is an isometry over $\mathcal{W}$.

B) Every function $W : \mathbb{R}^{L}_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{L}_{+}$, which is invariant with respect to every transformation $T_\Delta \in \mathcal{T}$, is constant over $\mathbb{R}^{L+1}_{++}$.

It is important to observe that $\mathcal{T}$ is just one of many possible classes of transformations which satisfy these two conditions, and our theory applies to any such class. However, what makes affine transformations special is its preservation of the possible rationality properties. It is staightforward to check that if a function w defined on $\mathbb{R}^{L}_{++} \times \mathbb{R}_{++}$ satisfies the weak axiom, then so will its transformation $T_\Delta[w]$, and if w is generated by the utility function $u(\cdot)$, then its transformation is generated by the utility function $u_\Delta = u(\Delta^{-1} \quad \cdot)$. This shows incidentally that it is possible, if one wishes so, to formulate all the assumptions put on the record in this paper on the (more fundamental ?) level of individual preferences, rather than on demand functions.

We shall make the following assumption (which characterizes the space $\mathcal{W}$ on which our result applies):

Assumption 1:

(i) The space $\mathcal{W}$ of admissible budget share functions is a subset of the set of all functions from $\mathbb{R}^{L}_{++} \times \mathbb{R}_{++}$ to $[0, \gamma]^L$ where $\gamma > 0$.

(ii) The normed subspace $\big(\mathcal{W}, \|\cdot\|_\infty\big)$ is compact.

(iii) $\mathcal{W}$ is large enough to verify:

$$\forall T_\Delta \in \mathcal{T}, \forall w \in \mathcal{W}, \quad T_\Delta[w] \in \mathcal{W}.$$

Compactness is the topological analogue of finiteness, and was already assumed by [13, p. 17], $\mathcal{M}_{\mathrm{com}}$. It can be thought of as arising from the continuity of some mapping that associates to each individual in, say, the real interval $[0,1]$ her budget share function. In other words, in a parametric setting, all we need is that the parameter set describing the set of feasible budget share functions be compact (see examples *infra*). Assumption (iii) corresponds to Assumption 1(2) in [24]. It requires the set of budget share functions to be large enough in order to remain stable by perturbations on prices and/or income. In particular, it prevents the set $\mathcal{W}$ from being finite, and we think of it as playing a role similar to the atomless hypothesis for "large" economies (see [18]).

We shall formalize a perfectly heterogenous population (in terms of households reactions to changes in prices and income) by an invariant measure with respect to every transformation $T_\Delta \in \mathcal{T}$. The following theorem establishes that this measure exists.

Theorem 1 *Under Assumption 1, there exists a (Borel-) probability measure λ on $\mathcal{W}$ such that the aggregate budget share function W is constant over $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$.*

Note that for a constant function W over $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$, one has: $\forall p, q \in \mathbb{R}^L_{++}$,

$$(p - q) \cdot \left(p^{-1} \quad W(p,x) - q^{-1} \quad W(q,x)\right) = -\sum_{l=1}^{L} \frac{(p_l - q_l)^2}{p_l q_l} W_l(p,x) \ \square \ 0. \tag{6}$$

Hence, from (4) we deduce that the Law of Demand holds in the aggregate. It is important to observe that, in contrast with [14] and [24], the Law of Demand is deduced here from the insensitivity property without any strong desirability requirement of any commodity. In particular, for any commodity, nothing requires the market budget share to be strictly positive for all prices. The following corollary extends Theorem 1 to a *finite* population not too far away from a perfectly heterogenous population.

Corollary 1 *Suppose that Assumption 1 is in force. For any $\varepsilon > 0$, there exists a probability distribution with finite support ν such that, for any $\Delta \gg 0$ and any $l \in \{1, \ldots, L\}$:*

$$\left| \int_{\mathcal{W}} w_l(\Delta \quad (p,x))\nu(dw) - \int_{\mathcal{W}} w_l(p,x)\nu(dw) \right| \ \square \ \varepsilon \quad (p,x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \tag{7}$$

If we further introduce the desirability requirement that for any given compact price set K, $W(p) > 0$, $\forall p \in K$, then we can deduce from Corollary 1 that the Law of Demand holds in K.[13] It is important to observe that, in contrast with [14] and [24], the latter assumption is not required for all prices but only for prices in K.

This approximate insensitivity of the market share should not be confused with the approximate results available in the literature. There, indeed, *atomless* economies are shown to have approximately insensitive market shares, and nothing is said about their finite approximations. Here, every finite economy which is approximately heterogenous exhibits an approximately constant market share, and converges, as the number of agents grows to infinity, towards a large economy that turns out to be perfectly heterogenous.

Notice that we nowhere assume that the budget share functions are continuous or homogeneous or that each individual budget constraint is satisfied.[14] Nor need the weak axiom of revealed preferences (WARP) be satisfied at any level[15] or the aggregate budget share function W be differentiable. This shows that extreme diversification of possibly extremely irregular and irrational characteristics may, on its own, generate an extremely regular mean outcome.

Is a measure like λ always immune against the criticism addressed by [3] and [4], [17] and [25] to the Grandmont-Quah-Kneip approach ? To make this point, it will suffice to show that any non-empty open subset of $\mathcal{W}$ is non-negligible.[16] For this purpose, the following additional assumption will fit the bill:

Assumption 2

For any pair $(v, w) \in \mathcal{W}, \quad \exists\, \Delta \in \mathbb{R}^L_{++} \quad / \quad w = T_\Delta[v]$.

Assumption 2 means that we restrict ourselves to the type of heterogeneity generated by the transformations T_Δ: it is possible to go from one's budget share function to another by composing transformations T_Δ. This requirement implies, as in [14], that all the individual budget share functions can be generated from a unique, fundamental one (the generator) by the class of transformations. We stress that this hypothesis is *not* needed for Theorem 1 to hold, hence to get the insensitivity of the aggregate budget share function. On the other hand, notice that, if the generator is distinct from a Cobb-Douglas function, so are *all* the individual budget share functions of the economy.

Proposition 1
Under Assumptions 1 and 2, the measure λ verifies:

$$\lambda(O) > 0 \quad \forall O \text{ non-empty, open subset of } \mathcal{W}.$$

By forbidding the concentration of the measure λ over any strict closed subset of $\mathcal{W}$, Assumptions 1 and 2 truly impose the behavioral heterogeneity we are looking for in this paper .

Proposition 1 prompts the question as to whether there is a *unique* way for the space of budget share functions $\mathcal{W}$ *to be perfectly heterogenously* distributed. The next result provides sufficient conditions on $\mathcal{W}$ for the measure λ to be unique. One could view it alternatively as 1) showing that 'behavioral heterogeneity' is defined in a non-ambiguous way; 2) suggesting that being heterogenous is a rather exceptional property for a population. In this context, it should be noted, however, that most of the micro-economic foundations of macro-economics we have in mind when dealing with the aggregation problem, as well as most of the econometric inquiries, do not need the population to be *perfectly* heterogenous. It usually suffices that it be sufficiently close to a 'uniform' distribution such as the one exhibited in the two preceding results. On the other hand, even in the atomless, perfectly heterogenous case, the measure λ is unique *given* some class $\mathcal{T}$ of transformations T_Δ. Changing this class would also change λ, so that the next result only shows a *conditional* uniqueness. With this in mind, 1) is probably the most relevant standpoint.

Assumption 3

(i) For any pair $T_\Delta, T_{\Delta'}$, if $T_\Delta[w] = T_{\Delta'}[w]\ \forall w$, then $\Delta = \Delta'$.

(ii) For any sequence $(\Delta_n)_n$ in $\left(\mathbb{R}^L_{++}\right)^{\mathbb{N}}$, and any continuous map $f : \mathcal{W} \to \mathcal{W}$, if $T_{\Delta_n}[w]$ converges to $f(w)$ uniformly on $\mathcal{W}$, then $\exists\ \Delta \gg 0$, such that $f(w) = T_\Delta[w],\ \forall w \in \mathcal{W}$.

Assumption 3 (i) suggests, roughly, that, for any price-income vector (p, x), whatever being the direction in which it is perturbed, there exist two consumers who react differently to this perturbation. It definitely rules out the degenerate case where all the individuals are Cobb-Douglas (in this case, indeed, there is no hope for getting uniqueness, since any probability measure would do the job). Assumption 3 (ii) is a technical, closedness requirement strengthening the compactness hypothesis 1(ii).

Theorem 2 *Under Assumptions 1 to 3, the 'uniform' probability distribution λ alluded to in Theorem 1 is unique.*

Is it possible to construct an example of non-trivial family of budget share functions, in such a way that the theory developed in this paper applies ? The following example answers positively to this question.

Example 1.
In the spirit of Grandmont's ([14]) seminal construction, our population is the collection of functions $\{w_\alpha\}_{\alpha\in\mathbb{R}^{L+1}}$ with w_α defined by:

$$w_\alpha(p, x) = T_{e^\alpha}[\overline{w}](p, x) := \overline{w}\big(e^{\alpha_1}p_1, e^{\alpha}_2 p_2, ..., e^{\alpha_{L+1}}x\big). \tag{8}$$

for all $(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, where $\overline{w}$ is a continuous function over $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$, and is called the generator. Notice that here, in contrast to [14], the distribution of the parameters α in the population is not assumed to admit a density function and *a fortiori* a flat density function.

The difficulty is that, as already pointed out, to ensure the compactness of $\mathcal{W}$, the parameter set has to be compact. Hence, one has to introduce an assumption that "compactifies" the parameter set $\mathbb{R}^{L+1}$. This assumption will be that $\overline{w}$ is completely described by its behavior over some compact subset of $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$. For this purpose, we introduce a second function $\tilde{w}$ from $\mathbb{R}^{L+1}$ into $\mathbb{R}^L_+$, defined by

$$\tilde{w}(t_1, t_2, ..., t_{L+1}) = \overline{w}(e^{t_1}, ..., e^{t_{L+1}}) \tag{9}$$

where $t = (t_1, t_2, ..., t_{L+1}) \in \mathbb{R}^{L+1}$, and let us begin with the simplest case, where heterogeneity of the households' share functions is required only with respect to one argument of the budget share function. This is the case, if for example, following [31], heterogeneity is to be required with respect to changes in income only. It implies that we can restrict ourselves to the subset of affine transformations such that $\alpha_i = 0$ for $i \square L$. We are thus concerned with the behavior of $\overline{w}$ over the space of positive income, $\mathbb{R}_{++}$. We simply make a periodicity requirement:

$$\tilde{w}(t + nc) = \tilde{w}(t), \tag{10}$$

for some $c > 0$. Under this assumption, the behavior of $\tilde{w}$ over $\mathbb{R}$ is entirely captured by its behavior over some compact interval $[k, k + c]$, where $k \in \mathbb{R}$. Moreover, its extreme points k and $k + c$ can be identified since $\tilde{w}$ assumes the same value for both. Hence, $\tilde{w}$ can be equivalently described by its behavior over the (compact) circle $\mathbb{R}/c\mathbb{Z}$. What kind of transformations are we going to use ? Obviously, some of them are now useless: we can content ourselves with translations of size less than c. Now, it is easy to see, using Ascoli's theorem, that the family of such transformed budget share functions will be relatively compact for the uniform topology. Taking its uniform closure yields compactness. Thus, according to Theorem 1, we can conclude that the economy just described admits a probability measure with respect to which it is perfectly heterogeneous. In order to see that our Assumption 2 is also satisfied, just observe that the set of translation parameters Δ itself is compact. Hence, if w belongs to the closure of $\mathcal{W}$, it must be the limit of some sequence $T_{\Delta_n}[\tilde{w}]$. It suffices to take the limit Δ^* of some subsequence of $(\Delta_n)_n$ to see that $w = T_{\Delta^*}[\tilde{w}]$. Hence, Proposition 1 holds. Finally, Assumption 3 is immediately satisfied if one adopts as space of translations the quotiented space $\mathbb{R}/c\mathbb{Z}$. As a consequence, Theorem 2 is verified.

This fairly simple example is analogous to, and can be compared with, the cases 2 and 3 in [33]. There, the population is described by a density function of the parameters α_{L+1} defined on $\mathbb{R}$. In this formalism, a heterogenous population is described by a flat density function over $\mathbb{R}$. As the density function becomes flatter, however, it has to be spreading to the left or to the right. This means that the values of α_{L+1} that predominate are those that are very small or very high. As already underlined by [17], this means that insensitivity at the aggregate might emerge because it is already satisfied at the individual level. Ruling out this trivial situation implies that for a fixed p, $\overline{w}(p, x)$ must have no limit as income goes to zero or infinity. This is also the case in our example just described. The difference between our approach and the one exemplified in [33], and followed by all the previous literature is that we can assert the existence of a perfectly heterogeneous probability measure, while in the already mentioned previous work, the exhibited distribution only induces some approximately heterogeneous measure. Moreover, thanks to our Corollary 1, we can assert that a perfectly heterogenous population can be approximated by a finite one, that is almost perfectly heterogenous. No such conclusion can be drawn from [33], as well as from the previous literature.

In order to extend the previous example to more sophisticated situations where $\tilde{w}$ depend upon more than one parameter, it suffices to consider the following periodicity requirement:

$$\tilde{w}(t + nc_i) = \tilde{w}(t) \quad \forall i = 1, ..., L + 1, \tag{11}$$

where $n \in \mathbb{Z}$, $c_i = \rho_i e_i \in \mathbb{R}^{L+1}_{++}$ with $\rho_i \in \mathbb{R}_{++}$ and $e_i = (0, ..., 1, 0...)$ denotes the i^{th} element of the canonical basis of $\mathbb{R}^{L+1}$ $\forall i$. Denote by $\mathcal{C}$ the compact set $\{t \in \mathbb{R}^{L+1} \ / \ t_i \in [0, \rho_i] \ \ \forall i = 1, ..., L+1\}$ and by $\mathcal{K}$ the compact set $\{(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \ / \ (p_1, ..., p_L, x) = (e^{t_1}, ..., e^{t_L}, e^x), \quad t \in \mathcal{C}\}$. Again, the function $\tilde{w}$ is entirely described by its behavior over $\mathcal{C}$ where, in addition, pieces of the boundary $\partial\mathcal{C}$ can be identified by taking the suitable quotients. Details are left to the reader. As outlined by [33], for many purposes (in particular for uniqueness and global stability of the price equilibrium) what we are interested in is the behavior of market demand on a compact set, K, of prices and income. Hence, it is enough to require that the household described by α has a budget share function which coincides with w_α only on K. In this case, the periodicity requirement of this example is not restrictive at all as long as $\mathcal{K}$ is chosen such that $K \subset \mathcal{K}$. In particular, our example does not require any "periodicity" of the generator over K, but only outside this relevant compact set. Furthermore, every function w^α in the population is of bounded variations over K.

Needless to say, several alternative examples could be constructed with various reparametrizations of the affine transformations.

Example 2.

If one is ready to give up the restriction to the class of affine transformations (which have the property to preserve the WARP), then several alternative examples can be easily constructed by using any class of transformations T_Δ fulfilling conditions A) and B). Here is the sketch of one possible construction, using rotations. To simplify the presentation we consider an economy with two commodities and we focus on heterogeneity of the households' share functions with respect to the price vector. All households possess the same income level, $x > 0$. The set of prices (p_1, p_2) can be identified with the non negative orthant of the complex plan:

$$\mathbb{C}_+ := \{z_p = p_1 + ip_2 \in \mathbb{C} \ : \ (p_1, p_2) \in \mathbb{R}^2_+\}. \tag{12}$$

Moreover prices are normalized so that the price space can be identified to $\mathbb{U}_+ := \mathbb{U} \cap \mathbb{C}_+$, where $\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$. This normalization can be justified by the assumption that households are not victims of money illusion (hence their budget share functions are homogeneous of degree zero in (p, x)). The population is the collection of functions $\{w_\Delta\}_{\Delta \in \mathbb{U}}$ with w_Δ defined by:

$$w_\Delta(p, x) = T_\Delta[\overline{w}](p, x) := \overline{w}(\Delta z_p, x), \tag{13}$$

where $\Delta \in \mathbb{U}$ is some unitary complex number (hence essentially acts as a rotation), while $\overline{w}$ is a continuous function over $\mathbb{C}_+ \times \mathbb{R}_{++}$, the generator. We assume that $\overline{w}$ satisfies the following property over $\mathbb{U}$: for all $n \in \mathbb{Z}$, one has

$$\overline{w}(e^{in\frac{\Pi}{2}} z_p, x) = \overline{w}(z_p, x). \tag{14}$$

In other words, any budget share function in the population is constant over the boundary of the price space, i.e.

$$w_\Delta(0,1,x) = w_\Delta(1,0,x). \tag{15}$$

We can therefore consider the equivalence relation

$$z \sim z' \Leftrightarrow \exists n \in \mathbb{Z} \quad / \quad z' = i^n z. \tag{16}$$

The quotiented space $\mathbb{U}_+/\sim$ (which can be identified to the price space) is denoted Π. It is not difficult to prove, along the same lines as in the previous example, that the population just defined (with the price space identified to Π) satisfies Assumptions 1 to 3. In this example, the only rationality required at the household level over the set of strictly positive prices is the continuity of the share function and the absence of money illusion. In particular, no periodicity assumption is required. The only restrictive assumption regards the boundary behavior of the household share function. Furthermore, if one is interested in the behavior of market demand on a compact set of prices, K, it is enough to require that the household described by Δ has a budget share function which coincides with w_Δ only on K. In this case, the latter requirement is harmless.

Remarks.

- According to the angle of attack adopted in this paper, three ingredients drive the insensitivity of aggregate budget share: (1) one needs a "large" population (this is Assumption 1(iii)), (2) whose characteristics are in a compact set (Assumptions 1(ii)) and (3) are uniformly distributed (this is the G-invariance of λ). For this uniformity requirement to describe a perfectly heterogenous population of households, we need the "specific type of heterogeneity" requirement (Assumption 2, Proposition 1). Finally, for it to be unambiguously defined, we need an additional assumption (Assumption 3(i)).

- In this paper, the 'uniform distribution' describes a population of households that are heterogenous in terms of their reaction to changes in prices *and* income. Note, however, that under the additional requirement that every individual budget share function is homogeneous of degree zero in (p,x) and satisfies the budget identity, one easily sees that Theorem 1 remains valid for the set of transformations T_Δ on the space $\mathcal{W}$ defined by perturbations of the price space solely, i.e. for the class of affine transformations,

$$\forall w \in \mathcal{W}, \forall \Delta \in \mathbb{R}^L_{++}, \forall (p,x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \quad T_\Delta[w](p,x) = w(\Delta \quad p, x). \tag{17}$$

- Suppose that income x at price p is defined by:

$$x := p \cdot \omega \tag{18}$$

where $\omega \in \mathbb{R}^L_+$ is the initial endowment in commodities of any household in our subpopulation. It is a routine matter to prove that, if the aggregate budget share function is insensitive to changes in prices and income, as it follows from our

Theorem 1, then market demand F satisfies the gross substitutability property. Indeed, consider two price systems p and q such that $q_l > p_l$ and $q_k = p_k$ for $k \neq l$. Denote by $F_k(p, p \cdot \omega)$ the market demand for commodity k at the price system p. The insensitivity property implies that

$$\frac{p_k F_k(p, p \cdot \omega)}{p \cdot \omega} = \frac{q_k F_k(q, p' \cdot \omega)}{q \cdot \omega}, \tag{19}$$

where by assumption $p \cdot \omega < q \cdot \omega$. Hence, for any pair $(p, q) \in (\mathbb{R}^L_{++})^2$ if $q_l > p_l$ and $q_k = p_k$ for $k \neq l$ one has

$$F_k(q, q \cdot \omega) > F_k(p, p \cdot \omega). \tag{20}$$

Thus, there is a unique equilibrium price, which is moreover globally stable in any standard *tâtonnement* process. Similarly, it is easy to prove that, if the aggregate budget share function is approximately insensitive to changes in prices and income, as it follows from our Corollary 1, then, for any given compact price set K and ε small enough, market demand F satisfies the gross substitutability property on K. As already said, under some standard, addititional assumption which guarantees that no equilibrium price exists outside the compact set of prices K, this again ensures uniqueness and global stability of the price equilibrium. Observe moreover, that nothing prevents from interpreting the collection $\{1, ..., L\}$ of "commodities" as composed of consumption goods and securities, possibly within an incomplete markets setting.

• In [21], the celebrated example of [2] is formaliwed by a "uniform" distribution over the space of individual characteristics which induces the insensitivity of the aggregate budget share function. The mathematical structure of this example is essentially the following: Consider the set of budget share functions $\mathcal{W} := \Sigma^{\mathbb{R}^{(L+1)}_{++}}$ as an uncountable product over the unit-simplex $\Sigma := \{x \in \mathbb{R}^L_+ : \sum_i x_i = 1\}$. Equip this space with the product topology. Thanks to Tychonov's theorem, $\mathcal{W}$ is compact. Kolmogorov's extension theorem insures furthermore that the infinite product of the Lebesgue measure $\lambda^{\mathbb{R}^{(L+1)}_{++}}$ is well-defined. John [21] proves that a population whose budget share functions are distributed according to the "uniform" probability $\lambda^{\mathbb{R}^{(L+1)}_{++}}$ has a market demand of the symmetric Cobb-Douglas type. The strength of this example is that, like in the general theory developed in this paper, no continuity assumption on the budget share functions is required. Since $\mathbb{R}^{(L+1)}_{++}$ is non-countable, however, the product topology is not metrizable, so that Theorem 1 does not apply to this setting. Moreover, it is difficult to think of any analogue of our Proposition 1 within John's framework.

3. Proofs of the results

The arguments involved are simple, and more or less familiar in measure theory. Here are nevertheless self-contained proofs.

Some preliminary remarks: It is clear (see [23] for details) that each transformation T_Δ is distance-preserving on the space $(\mathcal{W}, d)$, one-to-one and onto. Consider therefore the (Abelian) group G spanned by the transformations T_Δ. A generic element $g \in G$ is defined by:[17]

$$g = T_{\Delta_1} \circ ... \circ T_{\Delta_N} \quad \Delta_i \gg 0, \forall i. \tag{21}$$

By assumption 1 (ii), $\mathcal{W}$ is stable by any transformation T_Δ, thus, it must also be stable by the operation of the group G. Moreover, G operates isometrically on $\mathcal{W}$. Otherwise stated:

$$\forall g \in G, w, v \in \mathcal{W},\ gw \in W \text{ and } d(gw, gv) = d(w, v), \tag{22}$$

where d denotes the distance induced by the sup-norm.

3.1. Proof of Theorem 1

Since $\mathcal{W}$ is pre-compact with respect to d, for any $\varepsilon > 0$, there exists at least one finite subset $R(\varepsilon)$ of $\mathcal{W}$, such that, for any $w \in \mathcal{W}$, $\inf\{d(w, r) \ : \ r \in R(\varepsilon)\} \Box \varepsilon$. Let call $R(\varepsilon)$ a ε-network, and denote by $N(\varepsilon)$ the minimal cardinality of such ε-networks.

Claim. Let $\varepsilon > 0$, and R and R' two ε-networks of $\mathcal{W}$, of minimal cardinality $N(\varepsilon)$. There exists a bijection $\psi : R \to R'$, such that:

$$d(w, \psi(w)) \Box 2\varepsilon \quad \forall w \in \mathcal{W}. \tag{23}$$

To prove this claim, take $w \in R$, and consider the following set $A_w \subset R'$ of elements of R' which are "closely related" to w:

$$A_w = \{v \in R' \ : \ B(w, \varepsilon) \cap B(v, \varepsilon) \neq \emptyset\}. \tag{24}$$

Take, now, any subset $I \subset R$, and consider the set R'' obtained by replacing every element from I by the family of its "close" points:

$$R'' := (R \setminus I) \cup (\cup_{w \in I} A_w). \tag{25}$$

It is not difficult to see that R'' is still an ε-network of $\mathcal{W}$. Indeed, for any $x \in \mathcal{W}$, there exists some $w \in R$. If $w \notin I$, we are done. Otherwise, there must also exist some $v \in R'$ such that $d(v, x) \Box \varepsilon$. Hence $v \in A_w$, which implies that $v \in R''$.

Before going further, let us first recall the following well-known "wedding lemma" (see, for example, [15]):[19]

Lemma 1 *Let Y be a nonempty set, n some integer ≥ 1 and $A_1, ..., A_n$ be finite subsets of Y such that:*

$$\forall I \subset \{1, ..., n\}, \quad \#(\cup_{i \in I} A_i) \geq \#I. \tag{26}$$

Then, there exists a one-to-one mapping from I to $\prod_i A_i$.

In order to apply our wedding lemma, we need to verify:

$$\#R \; \square \; \#R'' \; \square \; \#(R \setminus I) + \#\big(\cup_{w \in I} A_w\big) \tag{27}$$

$$= \#R - \#I + \#\big(\cup_{w \in I} A_w\big). \tag{28}$$

This implies that $\#\big(\cup_{w \in I} A_w\big) \geq \#I$. Hence, there exists some one-to-one mapping $\psi : R \to \cup_{w \in R} A_w \subset R'$ such that $\psi(w) \in A_w$, $\forall w \in R$. Since $\#R = \#R'$, ψ is also onto. The inequality announced in the claim follows from the triangle inequality of the distance d.

In order to prove the theorem, take any sequence $(\varepsilon_n)_n$ of positive real numbers converging to 0, and, for every n, an ε_n-network R_n, of minimal cardinality $N(\varepsilon_n) = N_n$. Let us denote by:

$$\lambda_n := \frac{1}{n} \sum_{w \in R_n} \delta_w \tag{29}$$

the uniform probability measure over R_n. For any element $g \in G$, the finite network $R'_n := gR_n$ is still an ε-network of $\mathcal{W}$. Indeed, if $w \in \mathcal{W}$ and $x \in R_n$ such that $d(g^{-1}w, x) \; \square \; \varepsilon$, one has:

$$d(w, gx) = d(g^{-1}w, x) \; \square \; \varepsilon. \tag{30}$$

Take any bijection ψ as in the preceding claim, any function $F \in \mathcal{C}^0(\mathcal{W})$, and denote:

$$\alpha_n := \sup\big\{|F(w) - F(v)|, \; w, v \in \mathcal{W} \; / \; d(w, v) \; \square \; 2\varepsilon_n\big\}. \tag{31}$$

We have:

$$\int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} F(w)\lambda_n(dw) = \frac{1}{N_n}\Big[\sum_{w \in R_n} F(gw) - \sum_{w \in R_n} F(w)\Big] \tag{32}$$

$$= \frac{1}{N_n}\Big[\sum_{w \in R'_n} F(w) - \sum_{w \in R_n} F(w)\Big] = \frac{1}{N_n}\Big[\sum_{w \in R_n} (F(\psi(w) - F(w))\Big]. \tag{33}$$

It follows that:

$$|\int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} F(w)\lambda_n(dw)| = \frac{1}{N_n} \sum_{w \in R_n} |F(\psi(w) - F(w)| \; \square \; \alpha_n. \tag{34}$$

Banach-Alaoglu's theorem, again, implies that the sequence $(\lambda_n)_n$ of probability measures admits a subsequence that converges for the weak-$*$ topology to some probability measure, say, λ. On the other hand, since $\mathcal{W}$ is $\sigma(L_\infty, L_1)$-compact, F is uniformly continuous, so that $\alpha_n \to 0$ as n grows to infinity. Moreover, the mapping $w \mapsto F \circ g(w)$ is $\sigma(L_\infty, L_1)$-continuous. Hence, (34) yields, by passing to the limit:

$$\int_{\mathcal{W}} F(gw)\lambda(dw) = \int_{\mathcal{W}} F(w)\lambda(dw), \quad \forall g \in G \tag{35.}$$

In order to conclude the proof of the Theorem, consider the application $F_{p,x}$: $w \mapsto w(p,x)$. $F_{p,x} : \big(\mathcal{W}, d\big) \to \mathbb{R}$ is continuous. Thus, the preceding equality yields:

$$\int_{\mathcal{W}} w(\Delta \quad (p,x))d\lambda = \int_{\mathcal{W}} wd\lambda, \quad \forall p, x, \Delta \tag{36}$$

□

It should be noted that the measure λ is, in general, *not* the Haar measure of any (locally compact) group. What theorem 2 does is essentially to provide sufficient conditions ensuring that λ can be viewed as *the* Haar measure on the group G, and to take advantage from the uniqueness of this last measure.

3.2. Proof of Corollary 1

Since $\mathcal{W}$ is compact, the space of continuous functions on $\mathcal{W}$ is separable, i.e., admits a countable and dense subset $(f_n)_n$. Hence, the weak$-*$ topology on $\Delta(\mathcal{W})$ can be metrized by, e.g., the distance induced by the countable collection of semi-norms $p_f(\mu) := \int_{\mathcal{W}} f(w)\mu(dw)$:

$$d(\mu,\nu) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_{f_i}(\nu-\mu)}{1+p_{f_i}(\nu-\mu)}. \tag{37}$$

With the notations introduced in the proof of Theorem 1:

$$|\int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} F(w)\lambda_n(dw)| \;\square\; |\int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} f_i(gw)\lambda_n(dw)| +$$

$$|\int_{\mathcal{W}} f_i(gw)\lambda_n(dw) - \int_{\mathcal{W}} f_i(w)\lambda_n(dw)| + |\int_{\mathcal{W}} F(w)\lambda_n(dw) - \int_{\mathcal{W}} f_i(w)\lambda_n(dw)|. \tag{38}$$

For f_i $\frac{\varepsilon}{3}$-close to F, if $\alpha_n = \frac{\varepsilon}{3}$, this yields:

$$|\int_{\mathcal{W}} F(gw)d\lambda_n - \int_{\mathcal{W}} F(w)d\lambda_n| \;\square\; \varepsilon. \tag{39}$$

Hence, it suffices to take $\nu = \lambda_n$ for n large enough.

□

3.3. Proof of Proposition 1

It suffices to show that, for any $w \in \mathcal{W}$ and any $\varepsilon > 0$, there exists a collection $(g_1, ..., g_n) \in G^n$ such that

$$\mathcal{W} = \cup_{i=1}^{n} B(g_i w, \varepsilon). \tag{40}$$

This easily follows from the pre-compactness of $\mathcal{W}$ and assumption 2. In turn, (40) implies that each open ball $B(g_i w, \varepsilon)$ must be non-negligible with respect to λ. Indeed,

$$1 = \lambda(\mathcal{W}) \leq \textstyle\sum_i \lambda\big(B(g_i w, \varepsilon)\big) = \sum_i \lambda\big(g_i B(w, \varepsilon)\big)$$
$$= \sum_i \lambda\big(B(w,\varepsilon)\big) = n\lambda\big(B(w,\varepsilon)\big). \tag{41}$$

The first equality comes from the fact that G operates isometrically; the second from the G-invariance of λ.

□

3.4. Proof of Theorem 2

Thanks to assumption 3 (i), we can identify each element $g \in G$ with its (continuous) canonically associated mapping on $\mathcal{W}$, $\varphi_g : \mathcal{W} \to \mathcal{W}$:

$$\forall w \in \mathcal{W}, \quad \varphi_g(w) = gw. \tag{42}$$

On the other hand, let endow G with the following metric:

$$\delta(g,h) := \sup_{w \in \mathcal{W}} d(g(w), h(w)) \quad g, h \in G. \tag{43}$$

It easily follows that the family of mappings $\varphi_g : w \mapsto gw, \ g \in G$ is equi-continuous. Indeed, for any $\varepsilon > 0$, one has:

$$\forall w, v \in \mathcal{W}, \forall g \in G, \ d(w,v) \leq \varepsilon \Rightarrow d(g(w), g(v)) \leq \varepsilon. \tag{44}$$

Thanks to Ascoli's theorem, (G, δ) is therefore relatively compact. But assumption 3 (ii) says precisely that (G, δ) is closed. Hence, G is now a compact topological group. Consider the right-hand translation:

$$R_g(h) = hg \quad g, h \in G. \tag{45}$$

One has:

$$\delta(R_g(h) - R_g(h')) = \sup_w d(hg(w), h'g(w)) = \sup_w d(h(w), h'(w)) \tag{46}$$

because of assumption 2 and of the distance-preserving property of any g in G. Thus, $R_g(\cdot)$ is an isometry. We therefore can apply Theorem 1 on the group G itself, viewed as operating on itself *via* $R_g(\cdot)$. Thus, that there exists a probability μ on (G, d) verifying, for any continuous mapping $F : (G, d) \to (G, d)$:

$$\int_G F(hg)\mu(dh) = \int_G F(h)\mu(dh) \quad g \in G. \tag{47}$$

Obviously, μ is the Haar measure associated with (G, d). Let fix λ, a 'uniform distribution', $w \in \mathcal{W}$, $g \in G$ and $F \in \mathcal{C}^0\big(\mathcal{W}, \mathcal{W}\big)$. One has:

$$\int_{\mathcal{W}} F(w)\lambda(dw) = \int_{\mathcal{W}} F(gw)\lambda(dw). \tag{48}$$

Let us integrate both terms of the last equality with respect to $\mu(dg)$. Since F is continuous, hence bounded, Fubini's theorem yields:

$$\int_{\mathcal{W}} F(w)\lambda(dw) = \int_{\mathcal{W}} \lambda(dw)\Big[\int_G F(gw)\mu(dg)\Big]. \tag{49}$$

By assumption 2, for each w, there exists a h such that $w = hv$. Thus,

$$\int_G F(gw)\mu(dg) = \int_G F(ghv)\mu(dg) = \int_G F(gv)\mu(dg). \tag{50}$$

It follows that:

$$\int_{\mathcal{W}} F(w)\lambda(dw) = \int_{\mathcal{W}} \lambda(dw)\Big[\int_G F(gw)\mu(dg)\Big] = \int_G F(gw)\mu(dg). \tag{51}$$

Hence the result follows from the uniqueness of the Haar measure (see [7, chap. 7(1), Theorem 1, p.13]).

□

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Footnotes

2. Interestingly enough, an analogous conclusion has been drawn, for completely different reasons, in strategic game theory. Indeed, even the most demanding refinements of strategic stability lead to some intrinsically unavoidable indeterminacy of equilibria (see [29])). On the other hand, the aggregation problem faced here is definitely not to be confused with the *identification* issue, treated, e.g., in [11]. In the former case, one tries to deduce from micro-economic assumptions some sound restrictions on the macro-economic level; in the latter, one deduces the micro-economic characteristics of an economy from macro-economic observations. [34] is an excellent survey on all these issues.

3. In a parallel perspective, [13] showed that "dispersed" (non-smooth) preferences can imply a smooth aggregate demand, while the famous Lyapunov-Richter theorem says that convexity can be recovered from the non-atomicity of the measure space of agents.

4. Hildenbrand subsequently showed in [20] that an assumption over the distribution of individual demand vectors ensures the positive semidefiniteness in the aggregate of the income effect matrix. The Law of Demand follows then from the Slutsky decomposition of the Jacobian matrix of market demand. In this approach, individual rationality was still required to give an account of the negative semidefiniteness in the aggregate of the substitution effect matrix. Notice that [26] and [32] obtained the Law of Demand for more general income distributions. However, this was done at the cost of additional requirements on individual behaviors (or on the aggregate substitution effect matrix).

5. Some of the ideas related to heterogeneity have been also applied in [8] to financial asset economies with heterogenous beliefs, showing the versatility of this approach.

6. Note that two similar explanations of the Law of Demand were already proposed by [16, p.64] for the excess demand function : Hicks underlines, indeed, that the Law of Demand emerges in the aggregate if either the income effect is negligible at the micro-economic level or income effects *cancel out when aggregating* over buyers and sellers.

7. In this case Grandmont's formalism can be related to Hildenbrand and Kneip's ([21]) alternative formalization, see [33] for a discussion.

8. These terminologies follow respectively [5] and [25].

9. Note that this approximate insensitivity is sufficient to get the Law of Demand.

10. See also [35] and the references therein, especially [28].

11. On the other hand, our angle of attack is quite different: neither do we need to rely on the existence proof of a Haar measure on some locally compact

topological group in order to exhibit a 'uniform' distribution on agents' characteristics, nor do we restrict ourselves to individual preferences that are representable by smooth utility functions or to homogeneous budget constraints.

12. **Notations:** For any pair of vectors $x, y \in \mathbb{R}^L$, $x \cdot y$ denotes the Euclidean scalar product, and $x \quad y = (x_1 y_1, ..., x_L y_L)$ the tensor product. If $p \in \mathbb{R}^L_{++}$, p^{-1} denotes the vector $(\frac{1}{p_1}, ..., \frac{1}{p_L})$. For any bijective mapping $T : X \to X$ and any integer n, T^n stands for $T \circ ... \circ T$, the n^{th} composition of T with itself. Any Euclidean space is equipped with its Euclidean norm. $B(x, \varepsilon)$ is the open ball of center x and of radius ε. δ_x is the Dirac mass with support $\{x\}$; $\#X$ is the cardinality of the set X. For any topological space $X, \mathcal{C}^0(X)$ [resp. $L_\infty(X)$] is the space of continuous functions [resp. equivalence classes of bounded functions] $f : X \to X$.

13. This follows from the fact that $\forall p, q \in \mathbb{R}^L_{++}$,

$$(p-q) \cdot \Big(p^{-1} \quad W(p,x) - q^{-1} \quad W(q,x)\Big) \; \square \; (p-q)(p^{-1} - q^{-1}) \quad W(q) + (p-q)(p^{-1} - q^{-1}) \quad \varepsilon.$$

14. The last property would typically result from the local non-satiation of households' preferences, and would imply that $\sum_{l=1}^{L} w_l^i(p,x) = 1, \forall i, p, x$. This is required, e.g., in [14] and [31].

15. Observe that [14] and [24] do not need to assume the WARP at the individual level, while [31] does require it, because his assumptions on behavioral heterogeneity are weaker.

16. It is worth emphasizing that this property, while being sufficient, is not *necessary* to prove that an economy is heterogenous. The arguments provided in [33], for instance, are sufficient. Thus, the following property is a much stronger by-product.

17. Notice that $T_{-\Delta} = T_\Delta^{-1}$, $\forall \Delta$.

18. The interpretation should be clear: A_i is the set of boyfriends of Ms.i; if a certain collection of ladies I put their boyfriends in common, the number of men they get is at least as high as $\#I$. The conclusion of the lemma becomes then obvious: the n ladies will be able to marry without practicing polyandry.